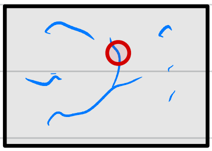
Descriptive Set Theory
Lecture 12

Banach Category Theorem. Let X be a top. space. If $A \subseteq X$ is locally meager, i.e. $\forall x \in A\ \exists$ open $U \ni x$ $U \cap A$ is meager, then A is meager.

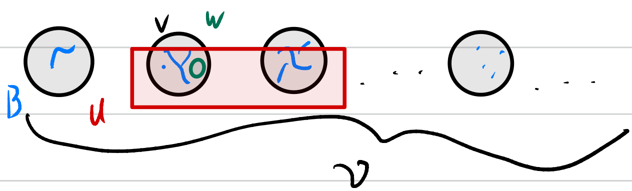


Proof. This is trivial for 2^{nd} ctbl spaces since $U_n \cap A$ meager for a ctbl cover (U_n) of A implies A is meager.

For the general X, observe the following:

Claim. If $\mathcal{V}$ is possibly unctbl collection of disjoint open sets that cover a set $B \subseteq X$ and $B \cap V$ is nowhere dense (resp. meager) for each $V \in \mathcal{V}$, then B is nowhere dense (resp. meager).

Proof.

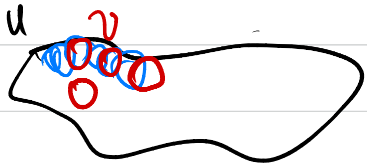


Let U be open s.t. $U \cap B \neq \emptyset$. Then $U \cap V \neq \emptyset$ for some $V \in \mathcal{V}$ and since $B \cap V$ is nowhere dense, $\exists\ \emptyset \neq W \subseteq V$ s.t. $B \cap W = (B \cap V) \cap W = \emptyset$.

The statement about meager follows from the one about nowhere dense. (Claim) $\square$

Now let $\mathcal{U}$ be the collection of all open sets U s.t. $U \cap A$ is meager. By the hypothesis, $U := \bigcup \mathcal{U} \supseteq A$.

By Zorn's lemma ($\Leftrightarrow$ AC), $\exists$ maximal disjoint subcollection $\mathcal{V} \subseteq \mathcal{U}$, so by the claim, $A \cap V$ is meager, where $V := \bigcup \mathcal{V}$. Let $U := \bigcup \mathcal{U}$. We'd like to show that $A \cap U = A$ is meager.



But observe that $U \setminus V$ is nowhere dense. Indeed, we show that V is dense in U, i.e. $U \subseteq \overline{V}$ so $U \setminus V \subseteq \overline{V} \setminus V = \partial V$, which is nowhere dense. Let $\emptyset \neq W \subseteq U$ be open. Then $\exists\, U' \in \mathcal{U}$ nonempty s.t. $U' \cap W =: W' \neq \emptyset$. But then if V and W' are disjoint, $\mathcal{V} \cup \{W'\}$ is a disjoint subcollection of $\mathcal{U}$, contradicting the maximality of $\mathcal{V}$. Thus, $V \cap W' \neq \emptyset$. $\square$

Now let $B \subseteq X$ in a top. space X. We want to understand when B is Baire measurable, in fact, distill a "largest part of B" that is Baire measurable.

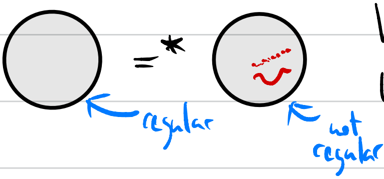
Let $\mathcal{U}(B) := \{U \subseteq X : U \text{ open and } U \Vdash B\}$, and $U(B) := \bigcup \mathcal{U}(B)$.

Prop. $\forall\, B \subseteq X$, $U(B) \Vdash B$, i.e. $U(B) \setminus B$ is meager. Moreover, B is Baire measurable $\Leftrightarrow$ $B \setminus U(B)$ is meager $\Leftrightarrow B =^* U(B)$.

Proof. $\mathcal{U}(B)$ is a cover of $U(B) \setminus B$, by def, so the Banach

category thm implies that $\mathcal{U}(B)\setminus B$ is meager.
Now if $B\setminus U(B)$ is meager, then $B =^* U(B)$.
And conversely, if B is Baire measurable, then $B =^* U$, so $U \in \mathcal{U}(B)$, hence $B \setminus U(B) \subseteq B \setminus U$ is meager. □

For any Baire meas. $B \subseteq X$, its $=^*$-class contains an open set by def., but many such open sets, e.g.



We call an open set *regular* if $U = \mathrm{int}(\overline{U})$. A closed set is *regular* if its complement is regular open.

For open $U \subseteq X$, let $\partial_{out} U := \{x \in \overline{U} : \forall \text{ open } V \ni x,\ V \not\subseteq \overline{U}\}$.

Prop. Let X be a top. space, $U \subseteq X$ and $F \subseteq X$ closed.
(a) U is regular $\Leftrightarrow$ $\partial U = \partial_{out} U$.
(b) F is regular $\Leftrightarrow$ $F = \overline{\mathrm{int}(F)}$.

Proof. Exercise.

Theorem. For any top. space X and $B \subseteq X$ Baire meas., $U(B)$ is the unique regular open set $=^* B$.
In particular, $B \mapsto U(B)$ is a "canonical selector"

for $BM(X)/_{=^*}$, i.e. selecting a set from each $=^*$-class.
In other words, $BM(X)/_{=^*} \cong RO(X) :=$ the collection of all regular open subsets of X.

Proof. Exercise.

The associated game. For a nonempty Polish space X, fix an ctbl open basis $\mathcal{U}$ and a complete compat. metric d.

For a set $B \subseteq X$, the **Banach–Mazur game** $G^{BM}(B)$ is

$$\begin{array}{llllll} \underline{P1.} & U_0 & & U_2 & & \dots \\ \underline{P2.} & & U_1 & & U_3 & \end{array}$$

s.t. $U_n \in \mathcal{U}$ is nonempty, $U_{n+1} \subseteq_c U_n :\Leftrightarrow \overline{U_{n+1}} \subseteq U_n$, and $\operatorname{diam}(U_n) < \frac{1}{2^n}$. P1 wins $:\Leftrightarrow \bigcap_n U_n = \{x\} \subseteq B$.

Theorem (Banach–Mazur, Oxtoby). Let X be nonempty Polish and $B \subseteq X$.

(a) Player 2 has a winning strategy in $G^{BM}(B) \Leftrightarrow B$ is meager.

(b) Player 1 has a winning strategy in $G^{BM}(B) \Leftrightarrow B$ is comeager in some nonempty open set.

Cor. If $G^{BM}(B \setminus U(B))$ is determined, then B is Baire measurable.
In particular, AD implies all subsets of Polish spaces are

Baire measurable.

Proof. Recall that B is Baire meas. $\Leftrightarrow$ $B \setminus U(B)$ is meager.

If P2 has a winning str. in $G^{BM}(B \setminus U(B))$, then $B \setminus U(B)$ is meager, so B is Baire meas.

We show that P1 cannot have a winning strategy in $G^{BM}(B \setminus U(B))$. Indeed, otherwise $\exists$ nonempty open $U \Vdash B \setminus U(B)$, so $U \in \mathcal{U}(B)$ contradicting $U \cap U(B) = \emptyset$. $\square$

Proof of Thm. (a) $\Leftarrow$. Suppose B is meager. Then $B \subseteq \bigcup_n F_n$ meager with F_n closed. When P1 has played U_{2n}, have P2 play any legal $U_{2n+1} \subseteq U_{2n} \setminus F_n$. (Note that $U_{2n} \setminus F_n$ is still nonmeager, hence nonempty.)

(a) $\Rightarrow$. Let σ be a winning str. for P2.

Call a position $p = (U_0, \dots, U_{2n+1})$ in σ good for a point $x \in X$ if $x \in U_{2n+1}$.

By convention, the empty position $p := ()$ is good for all points.

Note that $\forall x \in B$, $\exists$ maximal good position in σ (otherwise, the outcome of the game is $x \in B$, contradicting σ being a winning str. for P2). Thus, $B \subseteq \bigcup_{p \in \sigma} M_p$, where

$M_p := \{x \in X : p$ is maximal good for $x\}$. Since σ is ctbl, it remains to show that each M_p is nowhere dense. Let $U \neq \emptyset$ open and $p = (U_0, \dots, U_{2n+1})$. Let P1 play any legal $U_{2n+2} \subseteq U \cap U_{2n+1}$. If M_p is dense in U, then no matter what U_{2n+3} P2 plays according to σ, M_p would still intersect U_{2n+3}, contradicting the maximal goodness of p for the points in $M_p \cap U_{2n+3}$.

(b) <=>. Suppose $U \Vdash B$ and $U \neq \emptyset$. Then let $U_0 \subseteq U$ be any legal move for P1, so we still have $U_0 \Vdash B$.

P1	U_0		U_2		
P2		U_1		U_2	...

Then after U_0, the game is equivalent to P1 $\Leftrightarrow$ P2 swapped and B replaced with $U_0 \setminus B$, so part (a) implies (b). $\square$